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2005 J. Phys. A: Math. Gen. 38 9917

(http://iopscience.iop.org/0305-4470/38/46/002)

INSTITUTE OF PHYSICS PUBLISHING — JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 9917–9929 doi:10.1088/0305-4470/38/46/002

Generating functional analysis of CDMA detection dynamics

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Received 19 July 2005, in final form 7 September 2005
Published 2 November 2005
Online at stacks.iop.org/JPhysA/38/9917

Abstract

We investigate the detection dynamics of the parallel interference canceller (PIC) for code-division multiple-access (CDMA) multiuser detection, applied to a randomly spread, fully synchronous base-band uncoded CDMA channel model with additive white Gaussian noise (AWGN) under perfect power control in the large-system limit. It is known that the predictions of the density evolution (DE) can fairly explain the detection dynamics only in the case where the detection dynamics converge. At transients, though, the predictions of DE systematically deviate from computer simulation results. Furthermore, when the detection dynamics fail to converge, the deviation of the predictions of DE from the results of numerical experiments becomes large. As an alternative, generating functional analysis (GFA) can take into account the effect of the Onsager reaction term exactly and does not need the Gaussian assumption of the local field. We present GFA to evaluate the detection dynamics of PIC for CDMA multiuser detection. The predictions of GFA exhibit good consistency with the computer simulation result for any condition, even if the dynamics fail to converge.

PACS numbers: 89.70.+c, 05.45.−a, 05.50.+q

1. Introduction

Mobile communication systems, such as cellular phone systems, are now used every day by millions of people worldwide. Code-division multiple-access (CDMA) is a digital modulation system that employs spreading codes to enable access to a mobile communication system by multiple users [1]. In the multipoint-to-point communication framework, CDMA allows several users to share a single communication channel to a base station. Each user first modulates one's own information sequence using the spreading code assigned to the user, and then the modulated sequence is transmitted to the base station. The base station receives a mixture of the transmitted signals and additional channel noise. Using the users' spreading codes, a demodulator at the base station extracts the original information sequence from the received noise-degraded mixture signal. This process is called a detection.

Tanaka has evaluated the detection problem by the replica method [2–4]. However, the detection process cannot be treated by the replica method. The detection process of CDMA has drawn much attention from theoretical as well as practical viewpoints [5, 6]. Tanaka and Okada have applied a dynamical theory of Hopfield model [7] to the detection process [6]. Their method is equivalent to the density evolution (DE) framework in the field of information theory [8]. In the DE framework, a local field, which is a matched filter output that the estimated parallel interference is subtracted from, is separated into a signal part for the detection and a remaining noise part. Furthermore, it is assumed that the noise part follows a Gaussian distribution with mean zero. The predictions of DE can fairly explain the detection dynamics only in the case where the detection dynamics converge [6]. However, at transients the predictions of DE systematically deviate from computer simulation results. The Gaussian assumption of the local field has a more serious influence, when the detection dynamics fail to converge. In such a case, the deviation of the predictions of DE from the results of numerical experiments becomes large [6]. On the other hand, generating functional analysis (GFA) [9–12] does not need the Gaussian assumption. In this paper, we present GFA to evaluate the detection dynamics for CDMA multiuser detection, applied to a randomly spread, fully synchronous base-band uncoded CDMA channel model with additive white Gaussian noise (AWGN) under perfect power control. In order to confirm the validity of our analysis, we have performed computer simulations for some typical system load and channel noise conditions.

2. System model

We will focus on the basic fully synchronous K-user base-band binary phase-shift-keying (BPSK) CDMA channel model with perfect power control as

$$y^\mu \equiv \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_k^\mu b_k + \sigma_0 n^\mu, \tag{1}$$

where y^μ is the received signal at chip interval $\mu \in \{1, \ldots, N\}$, and where $b_k \in \{-1, 1\}$ and $s_k^\mu \in \{-1, 1\}$ are the BPSK-modulated information bit and the spreading code of user $k \in \{1, \ldots, K\}$ at chip interval μ, respectively. Figure 1 shows this CDMA communication model. The Gaussian random variable $\sigma_0^2 n^\mu$, where $n^\mu \sim N(0, 1)$, represents channel noise whose variance is σ_0^2. The spreading codes are independently generated from the identical unbiased distribution $P\left(s_k^\mu = 1\right) = P\left(s_k^\mu = -1\right) = 1/2$. The factor $1/\sqrt{N}$ is introduced in order to normalize the power per symbol to 1. Using these normalizations, the signal-to-noise ratio is defined as $Eb/N_0 = 1\big/\left(2\sigma_0^2\right)$. The ratio $\beta \equiv K/N$ is called system load.

The goal of multiuser detection is to simultaneously infer the information bits $b_1, \ldots, b_K$ after receiving the base-band signals $y_1, \ldots, y_N$. The updating rule for the tentative decision $\hat{b}_k(t) \in \{-1, 1\}$ of bit signal b_k at stage t is

$$\hat{b}_k(t) = \operatorname{sgn}\left(h_k - \sum_{k'=1, \neq k}^{K} W_{kk'} \hat{b}_{k'}(t-1)\right), \tag{2}$$

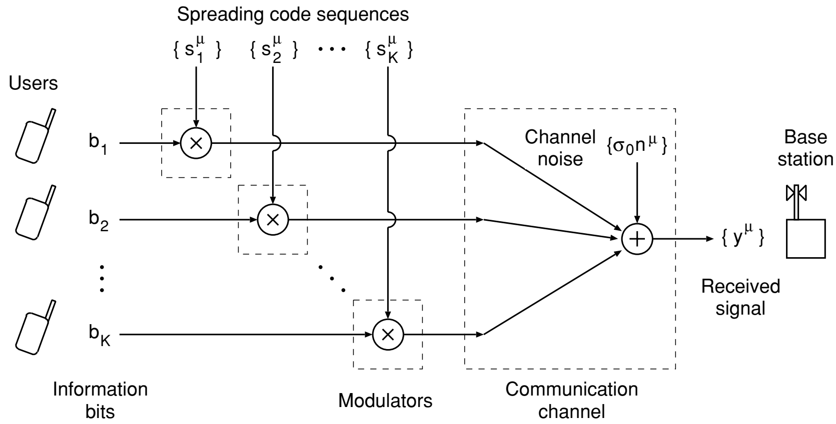

Figure 1. CDMA communication model.

where h_k is the output of the matched filter for user k:

$$h_k \equiv \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} s_k^\mu y^\mu, \tag{3}$$

and $W_{kk'}$ is the kk'-element of the sample correlation matrix $\boldsymbol{W}$ of the spreading code:

$$W_{kk'} \equiv \frac{1}{N} \sum_{\mu=1}^{N} s_k^\mu s_{k'}^\mu. \tag{4}$$

The function $\mathrm{sgn}(x)$ denotes the sign function taking 1 for $x \geqslant 0$ and -1 for $x < 0$. This iterative detection algorithm is called the parallel interference canceller (PIC) [1]. As for initialization, we assume the matched filter stage, i.e., $\hat{b}_k(0) = \mathrm{sgn}(h_k)$. This initialization is easily treated by formally assuming

$$\hat{b}_k(-1) = 0, \tag{5}$$

for all k. The widely used measure of the performance of a demodulator is the bit error rate (BER) $P_b(t)$, which is given by $P_b(t) = [1 - m(t)]/2$, where $m(t) = \frac{1}{K}\sum_{k=1}^{K} b_k \hat{b}_k(t)$ is the overlap between the information bits vector $\boldsymbol{b}(t) = {}^\dagger(b_1, \ldots, b_K)$ and the tentative decision vector $\hat{\boldsymbol{b}} = {}^\dagger(\hat{b}_1(t), \ldots, \hat{b}_K(t))$. The operator † denotes the transpose. Without loss of generality, we assume that the true information bits are all 1, i.e., $b_k = 1$ for all k, because the spreading codes are unbiased.

3. Generating functional analysis

3.1. Generating functional

We analyse the detection dynamics in the large-system limit where $K, N \to \infty$, while the system load β is kept finite. For generating functional analysis, we introduce inverse temperature γ. The stochastic updating rule for the tentative decision $\hat{b}_k(t) \in \{-1, 1\}$ of bit signal b_k at stage t is given by

$$P[\hat{b}_k(t+1) = -\hat{b}_k(t)] = \tfrac{1}{2}(1 - \tanh \gamma \hat{b}_k(t+1) u_k(t)), \tag{6}$$

where

$$u_k(t) \equiv h_k - \sum_{k'=1,\neq k}^{K} W_{kk'}\hat{b}_{k'}(t) + \theta_k(t), \tag{7}$$

which is called a local field. In the limit where $\gamma \to \infty$, this updating rule is equivalent to (2). The term $\theta_k(t)$ is a time-dependent external field which is introduced in order to define a response function. The inverse temperature and the external field are set $\gamma \to \infty$ and $\theta_k(t) = 0$ in the end of analysis.

To analyse the detection dynamics of the system we define a generating functional $Z[\boldsymbol{\psi}]$:

$$Z[\boldsymbol{\psi}] = \sum_{\hat{\boldsymbol{b}}(-1),\ldots,\hat{\boldsymbol{b}}(t)} p[\hat{\boldsymbol{b}}(-1),\ldots,\hat{\boldsymbol{b}}(t)] \exp\left(-\mathrm{i}\sum_{s=-1}^{t} \hat{\boldsymbol{b}}(s)\cdot\boldsymbol{\psi}(s)\right) \tag{8}$$

where $\hat{\boldsymbol{b}}(s) = {}^{\dagger}(\hat{b}_1(s),\ldots,\hat{b}_K(s))$, $\boldsymbol{\psi}(s) = {}^{\dagger}(\psi_1(s),\ldots,\psi_K(s))$. In a familiar way [9–12], one can obtain from $Z[\boldsymbol{\psi}]$ all averages of interest by differentiation, e.g.,

$$m_k(s) = \langle \hat{b}_k(s)\rangle = \mathrm{i}\lim_{\boldsymbol{\psi}\to\boldsymbol{0}} \frac{\partial Z[\boldsymbol{\psi}]}{\partial\psi_k(s)}, \tag{9}$$

$$C_{kk'}(s,s') = \langle \hat{b}_k(s)\hat{b}_{k'}(s')\rangle = -\lim_{\boldsymbol{\psi}\to\boldsymbol{0}} \frac{\partial Z[\boldsymbol{\psi}]}{\partial\psi_k(s)\partial\psi_{k'}(s')}, \tag{10}$$

$$G_{kk'}(s,s') = \frac{\partial\langle \hat{b}_k(s)\rangle}{\partial\theta_{k'}(s')} = \mathrm{i}\lim_{\boldsymbol{\psi}\to\boldsymbol{0}} \frac{\partial Z[\boldsymbol{\psi}]}{\partial\psi_k(s)\partial\theta_{k'}(s')}. \tag{11}$$

The dynamics (6) is a Markov chain, so the path probability $p[\hat{\boldsymbol{b}}(-1),\ldots,\hat{\boldsymbol{b}}(t)]$ is simply given by products of the individual transition probabilities $\rho[\hat{\boldsymbol{b}}(s+1)|\hat{\boldsymbol{b}}(s)]$ of the chain:

$$p[\hat{\boldsymbol{b}}(-1),\ldots,\hat{\boldsymbol{b}}(t)] = p[\hat{\boldsymbol{b}}(-1)]\prod_{s=-1}^{t-1}\rho[\hat{\boldsymbol{b}}(s+1)|\hat{\boldsymbol{b}}(s)], \tag{12}$$

where these transition probabilities are given by

$$\rho[\hat{\boldsymbol{b}}(s+1)|\hat{\boldsymbol{b}}(s)] = \prod_{k=1}^{K}\frac{\mathrm{e}^{\gamma\hat{b}_k(s+1)u_k(s)}}{2\cosh\gamma u_k(s)}. \tag{13}$$

Since the initial state is given by (5), the initial-state probability becomes $p[\hat{\boldsymbol{b}}(-1) = \boldsymbol{0}] = \prod_{k=1}^{K} p[\hat{b}_k(-1) = 0] = 1$. We separate the local field at any stage by inserting a following delta distributions:

$$1 = \int \delta\boldsymbol{u}\delta\hat{\boldsymbol{u}} \prod_{s=-1}^{t-1}\prod_{k=1}^{K}\exp\left(\mathrm{i}\hat{u}_k(s)\left[u_k(s) - h_k + \sum_{k'\neq k}^{K} W_{kk'}\hat{b}_{k'}(s) - \theta_k(s)\right]\right), \tag{14}$$

where $\delta\boldsymbol{u} \equiv \prod_{s=-1}^{t-1}\prod_{k=1}^{K}\frac{\mathrm{d}u_k(s)}{\sqrt{2\pi}}$ and $\delta\hat{\boldsymbol{u}} \equiv \prod_{s=-1}^{t-1}\prod_{k=1}^{K}\frac{\mathrm{d}\hat{u}_k(s)}{\sqrt{2\pi}}$. We can express (8) as

$$\begin{aligned} Z[\boldsymbol{\psi}] = \sum_{\hat{\boldsymbol{b}}(-1),\ldots,\hat{\boldsymbol{b}}(t)} & p[\hat{\boldsymbol{b}}(-1)]\int \delta\boldsymbol{u}\delta\hat{\boldsymbol{u}} \\ & \times\exp\left(\mathrm{i}\sum_{s=-1}^{t-1}\sum_{k=1}^{K}\hat{u}_k(s)\{u_k(s) - \hat{b}_k(s) - \theta_k(s)\} - \mathrm{i}\sum_{s=-1}^{t}\sum_{k=1}^{K}\hat{b}_k(s)\psi_k(s)\right) \\ & \times\exp\left(\sum_{s=0}^{t}\sum_{k=1}^{K}\{\gamma\hat{b}_k(s)u_k(s) - \ln 2\cosh\gamma u_k(s-1)\}\right) \end{aligned}$$

$$\times \exp\left(-\mathrm{i}\sqrt{\beta}\sigma_0 \sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\left[\frac{1}{\sqrt{K}}\sum_{k=1}^{K}\hat{u}_k(s)s_k^\mu\right]n^\mu\right)$$
$$\times \exp\left(-\mathrm{i}\beta \sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\left[\frac{1}{\sqrt{K}}\sum_{k=1}^{K}\hat{u}_k(s)s_k^\mu\right]\left[\frac{1}{\sqrt{K}}\sum_{k'=1}^{K}s_{k'}^\mu\{1-\hat{b}_{k'}(s)\}\right]\right). \quad (15)$$

In order to average the generating functional with respect to the disorder $\{s_k^\mu\}$ and $\{n^\mu\}$, we isolate the spreading codes by introducing the variables $v_\mu(s)$, $w_\mu(s)$:

$$1 = \int \delta\boldsymbol{v}\delta\hat{\boldsymbol{v}} \prod_{s=-1}^{t-1}\prod_{\mu=1}^{N}\exp\left(\mathrm{i}\hat{v}_\mu(s)\left[v_\mu(s) - \frac{1}{\sqrt{K}}\sum_{k=1}^{K}s_k^\mu\hat{u}_k(s)\right]\right), \quad (16)$$
$$1 = \int \delta\boldsymbol{w}\delta\hat{\boldsymbol{w}} \prod_{s=-1}^{t-1}\prod_{\mu=1}^{N}\exp\left(\mathrm{i}\hat{w}_\mu(s)\left[w_\mu(s) - \frac{1}{\sqrt{K}}\sum_{k=1}^{K}s_k^\mu\{1-\hat{b}_k(s)\}\right]\right), \quad (17)$$

where $\delta\boldsymbol{v} \equiv \prod_{\mu=1}^{N}\prod_{s=-1}^{t-1}\frac{\mathrm{d}v_\mu(s)}{\sqrt{2\pi}}$, $\delta\hat{\boldsymbol{v}} \equiv \prod_{\mu=1}^{N}\prod_{s=-1}^{t-1}\frac{\mathrm{d}\hat{v}_\mu(s)}{\sqrt{2\pi}}$, $\delta\boldsymbol{w} \equiv \prod_{\mu=1}^{N}\prod_{s=-1}^{t-1}\frac{\mathrm{d}w_\mu(s)}{\sqrt{2\pi}}$ and $\delta\hat{\boldsymbol{w}} \equiv \prod_{\mu=1}^{N}\prod_{s=-1}^{t-1}\frac{\mathrm{d}\hat{w}_\mu(s)}{\sqrt{2\pi}}$. The term in (15) containing the disorder becomes

$$\overline{\exp\left(-\mathrm{i}\sqrt{\beta}\sigma_0 \sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\left[\frac{1}{\sqrt{K}}\sum_{k=1}^{K}\hat{u}_k(s)s_k^\mu\right]n^\mu\right)}$$
$$\overline{\times \exp\left(-\mathrm{i}\beta \sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\left[\frac{1}{\sqrt{K}}\sum_{k=1}^{K}\hat{u}_k(s)s_k^\mu\right]\left[\frac{1}{\sqrt{K}}\sum_{k'=1}^{K}s_{k'}^\mu\{1-\hat{b}_{k'}(s)\}\right]\right)}$$
$$= \int \delta\boldsymbol{v}\delta\hat{\boldsymbol{v}}\delta\boldsymbol{w}\delta\hat{\boldsymbol{w}}\exp\left(\mathrm{i}\sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\{\hat{v}_\mu(s)v_\mu(s) + \hat{w}_\mu(s)w_\mu(s) - \beta v_\mu(s)w_\mu(s)\}\right)$$
$$\times \exp\left(-\frac{1}{2}\sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\sum_{s'=-1}^{t-1}\beta\sigma_0^2 v_\mu(s)v_\mu(s')\right)$$
$$\times \exp\left(-\frac{1}{2}\sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\sum_{s'=-1}^{t-1}\hat{v}_\mu(s)\left[\frac{1}{K}\sum_{k=1}^{K}\hat{u}_k(s)\hat{u}_k(s')\right]\hat{v}_\mu(s')\right)$$
$$\times \exp\left(-\frac{1}{2}\sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\sum_{s'=-1}^{t-1}\hat{v}_\mu(s)\left[\frac{1}{K}\sum_{k=1}^{K}\hat{u}_k(s) - \frac{1}{K}\sum_{k=1}^{K}\hat{u}_k(s)\hat{b}_k(s')\right]\hat{w}_\mu(s')\right)$$
$$\times \exp\left(-\frac{1}{2}\sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\sum_{s'=-1}^{t-1}\hat{w}_\mu(s)\left[\frac{1}{K}\sum_{k=1}^{K}\hat{u}_k(s') - \frac{1}{K}\sum_{k=1}^{K}\hat{u}_k(s')\hat{b}_k(s)\right]\hat{v}_\mu(s')\right)$$
$$\times \exp\left(-\frac{1}{2}\sum_{\mu=1}^{N}\sum_{s=-1}^{t-1}\sum_{s'=-1}^{t-1}\hat{w}_\mu(s)\left[1 - \frac{1}{K}\sum_{k=1}^{K}\hat{b}_k(s) - \frac{1}{K}\sum_{k=1}^{K}\hat{b}_k(s')\right.\right.$$
$$\left.\left. - \frac{1}{K}\sum_{k=1}^{K}\hat{b}_k(s)\hat{b}_k(s')\right]\hat{w}_\mu(s')\right), \quad (18)$$

where $\overline{\cdots}$ denotes averaging over the disorder $\left\{s_k^\mu\right\}$ and $\{n^\mu\}$. We separate the relevant one-stage and two-stage order parameters by inserting

$$1 = \left(\frac{K}{2\pi}\right)^{t+1} \int \mathrm{d}\boldsymbol{m}\,\mathrm{d}\hat{\boldsymbol{m}} \exp\left(\mathrm{i}K \sum_{s=-1}^{t-1} \hat{m}(s) \left[m(s) - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \hat{b}_k(s)\right]\right), \tag{19}$$

$$1 = \left(\frac{K}{2\pi}\right)^{t+1} \int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\hat{\boldsymbol{k}} \exp\left(\mathrm{i}K \sum_{s=-1}^{t-1} \hat{k}(s) \left[k(s) - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \hat{u}_k(s)\right]\right), \tag{20}$$

$$1 = \left(\frac{K}{2\pi}\right)^{(t+1)^2} \int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\hat{\boldsymbol{q}} \exp\left(\mathrm{i}K \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \hat{q}(s,s') \left[q(s,s') - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \hat{b}_k(s)\hat{b}_k(s')\right]\right), \tag{21}$$

$$1 = \left(\frac{K}{2\pi}\right)^{(t+1)^2} \int \mathrm{d}\boldsymbol{Q}\,\mathrm{d}\hat{\boldsymbol{Q}} \exp\left(\mathrm{i}K \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \hat{Q}(s,s') \left[Q(s,s') - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \hat{u}_k(s)\hat{u}_k(s')\right]\right), \tag{22}$$

$$1 = \left(\frac{K}{2\pi}\right)^{(t+1)^2} \int \mathrm{d}\boldsymbol{L}\,\mathrm{d}\hat{\boldsymbol{L}} \exp\left(\mathrm{i}K \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \hat{L}(s,s') \left[L(s,s') - \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \hat{b}_k(s)\hat{u}_k(s')\right]\right). \tag{23}$$

Since the initial-state probability is factorizable, the disorder-averaged generating functional factorizes into single-site contributions. The disorder-averaged generating functional is for $K \to \infty$ dominated by a saddle point. We can thus simplify the saddle-point problem to

$$\bar{Z}[\psi] = \int \mathrm{d}\boldsymbol{m}\,\mathrm{d}\hat{\boldsymbol{m}}\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\hat{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\hat{\boldsymbol{q}}\,\mathrm{d}\boldsymbol{Q}\,\mathrm{d}\hat{\boldsymbol{Q}}\,\mathrm{d}\boldsymbol{L}\,\mathrm{d}\hat{\boldsymbol{L}}\; \exp(K(\Psi + \Phi + \Omega) + O(\ln K)) \tag{24}$$

in which the functions Ψ, Φ, Ω are given by

$$\begin{aligned}\Psi \equiv{}& \mathrm{i} \sum_{s=-1}^{t-1} \{\hat{m}(s)m(s) + \hat{k}(s)k(s)\} \\ &+ \mathrm{i} \sum_{s=-1}^{t-1} \sum_{s'=0}^{t-1} \{\hat{q}(s,s')q(s,s') + \hat{Q}(s,s')Q(s,s') + \hat{L}(s,s')L(s,s')\}\end{aligned} \tag{25}$$

$$\begin{aligned}\Phi \equiv{}& \frac{1}{K}\sum_{k=1}^{K} \ln\Bigg\{ \sum_{\hat{b}(-1),\ldots,\hat{b}(t)} p[\hat{b}(-1)] \int \delta u \delta\hat{u} \exp\left(\sum_{s=0}^{t} \{\gamma \hat{b}(s) u(s-1) - \ln 2\cosh \gamma u(s-1)\}\right) \\ &\times \exp\left(-\mathrm{i} \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \{\hat{q}(s,s')\hat{b}(s)\hat{b}(s') + \hat{Q}(s,s')\hat{u}(s)\hat{u}(s') + \hat{L}(s,s')\hat{b}(s)\hat{u}(s')\}\right) \\ &\times \exp\left(\mathrm{i} \sum_{s=-1}^{t-1} \hat{u}(s)\{u(s) - \hat{b}(s) - \theta_k(s) - \hat{k}(s)\} - \mathrm{i} \sum_{s=-1}^{t-1} \hat{b}(s)\hat{m}(s) - \mathrm{i} \sum_{s=-1}^{t} \hat{b}(s)\psi_k(s)\right)\Bigg\}\end{aligned} \tag{26}$$

$$\Omega \equiv \frac{1}{K} \ln \int \delta \boldsymbol{v} \delta \hat{\boldsymbol{v}} \delta \boldsymbol{w} \delta \hat{\boldsymbol{w}} \exp \left(\mathrm{i} \sum_{\mu=1}^{N} \sum_{s=-1}^{t-1} \{ \hat{v}_\mu(s) v_\mu(s) + \hat{w}_\mu(s) w_\mu(s) - \beta v_\mu(s) w_\mu(s) \} \right)$$
$$\times \exp \left(-\frac{1}{2} \sum_{\mu=1}^{N} \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \left\{ \beta \sigma_0^2 v_\mu(s) v_\mu(s') + \hat{v}_\mu(s) Q(s,s') \hat{v}_\mu(s') \right\} \right)$$
$$\times \exp \left(-\frac{1}{2} \sum_{\mu=1}^{N} \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \{ \hat{v}_\mu(s) [k(s) - L(s',s)] \hat{w}_\mu(s') + \hat{w}_\mu(s) [k(s') - L(s,s')] \hat{v}_\mu(s') \} \right)$$
$$\times \exp \left(-\frac{1}{2} \sum_{\mu=1}^{N} \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \{ \hat{w}_\mu(s) [1 - m(s) - m(s') + q(s,s')] \hat{w}_\mu(s') \} \right) \tag{27}$$

where $\delta u \equiv \prod_{s=0}^{t-1} \frac{\mathrm{d}u(s)}{\sqrt{2\pi}}$ and $\delta \hat{u} \equiv \prod_{s=0}^{t-1} \frac{\mathrm{d}\hat{u}(s)}{\sqrt{2\pi}}$. We have arrived at a single-site saddle-point problem. Using normalization condition and $\bar{Z}[\mathbf{0}] = 1$, we obtain field derivatives of the generating functional as follows:

$$\overline{\langle \hat{b}_k(s) \rangle} = \langle \hat{b}(s) \rangle_*, \tag{28}$$

$$\overline{\langle \hat{b}_k(s) \hat{b}_{k'}(s') \rangle} = \delta_{k,k'} \langle \hat{b}(s) \hat{b}(s') \rangle_* + (1 - \delta_{k,k'}) \langle \hat{b}(s) \rangle_* \langle \hat{b}(s') \rangle_*, \tag{29}$$

$$\frac{\partial}{\partial \theta_{k'}(s')} \overline{\langle \hat{b}_k(s) \rangle} = -\mathrm{i} \delta_{k,k'} \langle \hat{b}(s) \hat{u}(s') \rangle_*, \tag{30}$$

where $\delta_{k,k'}$ is Kronecker's delta taking 1 if $k = k'$ and 0 otherwise and $\langle \cdot \rangle_*$ denotes

$$\langle f(\{\hat{b}, u, \hat{u}\}) \rangle_* \equiv \frac{\sum_{\hat{b}(-1),\ldots,\hat{b}(t)} \int \delta u \delta \hat{u} M(\{\hat{b}, u, \hat{u}\}) f(\{\hat{b}, u, \hat{u}\})}{\sum_{\hat{b}(-1),\ldots,\hat{b}(t)} \int \delta u \delta \hat{u} M(\{\hat{b}, u, \hat{u}\})}, \tag{31}$$

with

$$M(\{\hat{b}, u, \hat{u}\}) \equiv p[\hat{b}(-1)] \exp \left(\sum_{s=0}^{t} \{ \gamma \hat{b}(s) u(s-1) - \ln 2 \cosh \gamma u(s-1) \} \right)$$
$$\times \exp \left(-\mathrm{i} \sum_{s=-1}^{t-1} \sum_{s'=-1}^{t-1} \{ \hat{q}(s,s') \hat{b}(s) \hat{b}(s') + \hat{Q}(s,s') \hat{u}(s) \hat{u}(s') + \hat{L}(s,s') \hat{b}(s) \hat{u}(s') \} \right)$$
$$\times \exp \left(\mathrm{i} \sum_{s=-1}^{t-1} \hat{u}(s) \{ u(s) - \hat{b}(s) - \theta_k(s) - \hat{k}(s) \} - \mathrm{i} \sum_{s=-1}^{t-1} \hat{b}(s) \hat{m}(s) \right) \Bigg|_{\mathrm{saddle}}. \tag{32}$$

The evaluation $f|_{\mathrm{saddle}}$ denotes an evaluation of a function f at the dominating saddle point. Therefore, we see the order parameters are essentially single-site ones.

3.2. saddle-point equations

In the limit $K \to \infty$, the integral (24) will be dominated by the saddle point of the extensive exponent $\Psi + \Phi + \Omega$. We first calculate the remaining Gaussian integral in Ω:

$$\Omega = \frac{1}{\beta}\int \frac{\mathrm{d}\hat{\boldsymbol{v}}}{(2\pi)^{(t+1)/2}}\frac{\mathrm{d}\hat{\boldsymbol{w}}}{(2\pi)^{(t+1)/2}}$$
$$\times \exp\left(\mathrm{i}^{\dagger}\hat{\boldsymbol{w}}(\beta^{-1}\mathbf{1})\hat{\boldsymbol{v}} - \frac{1}{2}{}^{\dagger}\hat{\boldsymbol{v}}\boldsymbol{Q}\hat{\boldsymbol{v}} - \frac{1}{2}{}^{\dagger}\hat{\boldsymbol{v}}^{\dagger}\boldsymbol{B}\hat{\boldsymbol{w}} - \frac{1}{2}{}^{\dagger}\hat{\boldsymbol{w}}\boldsymbol{B}\hat{\boldsymbol{v}} - \frac{1}{2}{}^{\dagger}\hat{\boldsymbol{w}}\hat{\boldsymbol{D}}\hat{\boldsymbol{w}}\right)$$
$$= \frac{1}{\beta}\int \frac{\mathrm{d}\hat{\boldsymbol{v}}}{(2\pi)^{t/2}}\exp\left(-\frac{1}{2}{}^{\dagger}\hat{\boldsymbol{v}}\boldsymbol{Q}\hat{\boldsymbol{v}}\right)|\hat{\boldsymbol{D}}|^{-1/2}\exp\left(-\frac{1}{2}{}^{\dagger}\hat{\boldsymbol{v}}^{\dagger}(\beta^{-1}\mathbf{1}-\boldsymbol{B})\hat{\boldsymbol{D}}^{-1}(\beta^{-1}\mathbf{1}-\boldsymbol{B})\hat{\boldsymbol{v}}\right)$$
$$= -\frac{1}{2\beta}(\ln|\hat{\boldsymbol{D}}| + \ln|\boldsymbol{Q} + {}^{\dagger}(\beta^{-1}\mathbf{1}-\boldsymbol{B})\hat{\boldsymbol{D}}^{-1}(\beta^{-1}\mathbf{1}-\boldsymbol{B})|), \tag{33}$$

where $\boldsymbol{B}$, $\hat{\boldsymbol{D}}$ and $\boldsymbol{Q}$ are matrices having matrix elements

$$B(s,s') \equiv -\mathrm{i}k(s') - G(s,s'), \tag{34}$$

$$\hat{D}(s,s') \equiv \frac{\sigma_0^2}{\beta} + 1 - m(s) - m(s') + C(s,s'), \tag{35}$$

and $Q(s,s')$, respectively. The saddle-point equations are derived by differentiation with respect to integration variables $\{\boldsymbol{m}, \hat{\boldsymbol{m}}, \boldsymbol{k}, \hat{\boldsymbol{k}}, \boldsymbol{q}, \hat{\boldsymbol{q}}, \boldsymbol{Q}, \hat{\boldsymbol{Q}}, \boldsymbol{L}, \hat{\boldsymbol{L}}\}$. These equations will involve the average single-site correlation $C(s,s')$ and the average single-site response function $G(s,s')$:

$$C(s,s') = \lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\overline{\langle\hat{b}_k(s)\hat{b}_{k'}(s')\rangle} = \langle\hat{b}(s)\hat{b}(s')\rangle_*, \tag{36}$$

$$G(s,s') = \lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\frac{\partial}{\partial\theta_{k'}(s')}\overline{\langle\hat{b}_k(s)\rangle} = -\mathrm{i}\langle\hat{b}(s)\hat{u}(s')\rangle_*. \tag{37}$$

Straightforward differentiation by usage of causality leads us to the following saddle-point equations:

$$\hat{m}(s) = k(s) = \hat{q}(s,s') = Q(s,s') = 0, \tag{38}$$

$$\hat{k}(s) = |\boldsymbol{\Lambda}_s|, \tag{39}$$

$$m(s) = \langle\hat{b}(s)\rangle_*, \tag{40}$$

$$q(s,s') = \langle\hat{b}(s)\hat{b}(s')\rangle_* = C(s,s'), \tag{41}$$

$$L(s,s') = \mathrm{i}G(s,s') = \begin{cases}-\mathrm{i}\langle\hat{b}(s)\hat{u}(s')\rangle_*, & \text{for } s > s'\\ 0, & \text{for } s \leqslant s',\end{cases} \tag{42}$$

$$\hat{\boldsymbol{Q}} = -\mathrm{i}\tfrac{1}{2}{}^{\dagger}(\mathbf{1}+\beta\boldsymbol{G})^{-1}\boldsymbol{D}(\mathbf{1}+\beta^{\dagger}\boldsymbol{G})^{-1}, \tag{43}$$

$$\hat{\boldsymbol{L}} = (\mathbf{1}-\beta^{\dagger}\boldsymbol{G})^{-1}, \tag{44}$$

where $\hat{\boldsymbol{Q}}$, $\hat{\boldsymbol{L}}$, $\boldsymbol{D}$ and $\boldsymbol{\Lambda}_s$ are matrices having matrix elements $\hat{Q}(s,s')$, $\hat{L}(s,s')$,

$$D(s,s') \equiv \beta\hat{D}(s,s') = \sigma_0^2 + \beta[1 - m(s) - m(s') + C(s,s')], \tag{45}$$

and

$$\Lambda_s(s',s'') \equiv \begin{cases}\delta_{s',s''} + \beta G(s'',s'), & \text{for } s' \neq s\\ 1, & \text{for } s' = s,\end{cases} \tag{46}$$

respectively. Substituting (38)–(44) into (31) and introducing a simple rescaling of local fields and conjugate local fields, the term $\langle \hat{b}(s)\hat{\boldsymbol{u}}\rangle_*$ becomes

$$\langle \hat{b}(s)\hat{\boldsymbol{u}}\rangle_* = \int \delta u \delta \hat{u} \sum_{\hat{b}(-1),\ldots,\hat{b}(t)} \hat{b}(s)\hat{\boldsymbol{u}} \exp\left(\sum_{s=-1}^{t-1}\{\gamma \hat{b}(s+1)u(s) - \ln 2\cosh \gamma u(s)\}\right)$$
$$\times \exp\left(-\frac{1}{2}{}^{\dagger}\hat{\boldsymbol{u}}\boldsymbol{R}\hat{\boldsymbol{u}} + \mathrm{i}\hat{\boldsymbol{u}}(\boldsymbol{u} - \hat{\boldsymbol{k}} - \boldsymbol{\theta} - \boldsymbol{\Gamma}\hat{\boldsymbol{b}})\right)$$
$$= \mathrm{i}\int \frac{\mathrm{d}\boldsymbol{v} \exp\left(-\frac{1}{2}\boldsymbol{v}\cdot R^{-1}\boldsymbol{v}\right)}{\sqrt{|2\pi \boldsymbol{R}|}} \sum_{\hat{b}(-1),\ldots,\hat{b}(t)} \hat{b}(s)\boldsymbol{R}^{-1}\boldsymbol{v} \prod_{s=-1}^{t-1} \frac{1}{2}[1 + \hat{b}(s+1)\,\mathrm{sgn}\,u(s)], \tag{47}$$

in the limit $\gamma \to \infty$, where $\hat{\boldsymbol{u}} \equiv {}^{\dagger}(\hat{u}(-1), \ldots, \hat{u}(t-1))$, $\boldsymbol{R} \equiv (\mathbf{1}+\beta^{\dagger}\boldsymbol{G})^{-1}\boldsymbol{D}(\mathbf{1}+\beta\boldsymbol{G})^{-1}$ and $\boldsymbol{\Gamma} \equiv (\mathbf{1}+\beta\boldsymbol{G})^{-1}\beta\boldsymbol{G}$. The terms $\langle \hat{b}(s)\rangle_*$ and $\langle \hat{b}(s)\hat{b}(s')\rangle_*$ can also be calculated in a similar way.

Let us summarize our calculation. Some macroscopic integration variables are found to vanish in the relevant physical saddle point: $\hat{m(s)} = k(s) = \hat{q}(s,s') = Q(s,s') = 0$. The remaining ones can all be expressed in terms of three macroscopic observables, namely the overlaps $m(s)$, the single-site correlation functions $C(s,s')$ and the single-site response functions $G(s,s')$. Finally, setting $\gamma \to \infty$ and $\theta(s) = 0$, we then arrive at the following saddle-point equations in the thermodynamic limit, i.e., $K \to \infty$:

$$m(s) = \langle\langle \hat{b}(s)\rangle\rangle, \tag{48}$$

$$C(s,s') = \langle\langle \hat{b}(s)\hat{b}(s')\rangle\rangle, \tag{49}$$

$$G(s,s') = \begin{cases} \langle\langle \hat{b}(s)(\boldsymbol{R}^{-1}\boldsymbol{v})(s')\rangle\rangle, & \text{for } s > s' \\ 0, & \text{for } s \leqslant s'. \end{cases} \tag{50}$$

The bit error rate is obtained by

$$P_b(s) = \frac{1 - m(s)}{2}. \tag{51}$$

The average over the effective path measure is given by

$$\langle\langle g(\hat{\boldsymbol{b}, \boldsymbol{v}})\rangle\rangle \equiv \int \mathcal{D}\boldsymbol{v}\, \mathrm{Tr}\, g(\hat{\boldsymbol{b}}, \boldsymbol{v}) \prod_{s=-1}^{t-1} \frac{1}{2}[1 + \hat{b}(s+1)\,\mathrm{sgn}\,u(s)], \tag{52}$$

$$\mathcal{D}\boldsymbol{v} \equiv \frac{\mathrm{d}\boldsymbol{v}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{R}^{-1}\boldsymbol{v}}}{\sqrt{|2\pi \boldsymbol{R}|}}, \tag{53}$$

$$\mathrm{Tr} \equiv \sum_{\hat{b}(-1)\in\{0\},\hat{b}(0),\ldots,\hat{b}(t)\in\{-1,1\}}, \tag{54}$$

$$u(s) = \hat{k}(s) + v(s) + (\boldsymbol{\Gamma}\hat{\boldsymbol{b}})(s), \tag{55}$$

$$\boldsymbol{R} = (\mathbf{1}+\beta^{\dagger}\boldsymbol{G})^{-1}\boldsymbol{D}(\mathbf{1}+\beta\boldsymbol{G})^{-1}, \tag{56}$$

$$\boldsymbol{\Gamma} = (\mathbf{1}+\beta\boldsymbol{G})^{-1}\beta\boldsymbol{G}, \tag{57}$$

$$\hat{k}(s) = |\boldsymbol{\Lambda}_s|, \tag{58}$$

$$D(s,s') \equiv \sigma_0^2 + \beta[1 - m(s) - m(s') + C(s,s')], \tag{59}$$

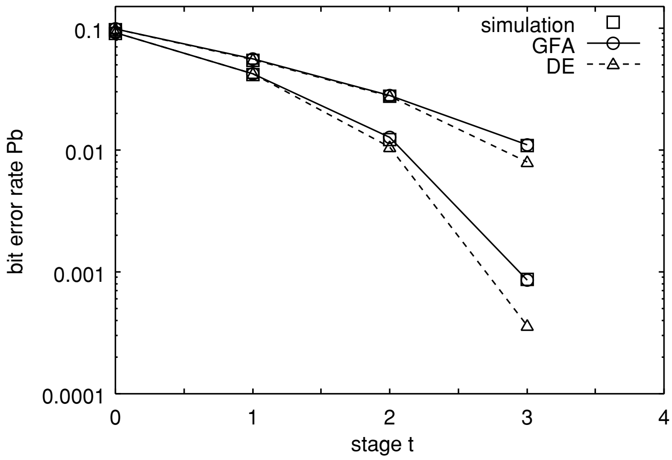

Figure 2. The first few stages of the detection dynamics predicted by generating functional analysis (solid line) and density evolution (dashed line). Computer simulations (square) are evaluated with $N = 8000$ from 100 experiments for the cases $E_b/N_0 = 7.0$ (dB) (upper), 9.0 (dB) (lower). The system load is $\beta = 0.5 < \beta_c$ for both cases.

$$\Lambda_s(s', s'') = \begin{cases} \delta_{s',s''} + \beta G(s'', s'), & \text{for } s' \neq s \\ 1, & \text{for } s' = s. \end{cases} \tag{60}$$

The terms $(\boldsymbol{R}^{-1}\boldsymbol{v})(s)$ and $(\boldsymbol{\Gamma\sigma})(s)$ denote the sth element of the vector $\boldsymbol{R}^{-1}\boldsymbol{v}$ and $\boldsymbol{\Gamma\sigma}$, respectively. Equations (48)–(60) entirely describe the dynamics of the system. In the limit where $t \to \infty$, the term $(\boldsymbol{\Gamma\sigma})(s)$ in (55) can be regarded as a self-interaction and corresponds to the Onsager reaction term in equilibrium statistical mechanics. Therefore, in this paper, we call this term the Onsager reaction term.

4. Results and discussion

In order to validate the results obtained above, we performed numerical experiments in an $N = 8000$ system. Figure 2 shows the first few stages of the detection dynamics obtained from 100 experiments for the cases $E_b/N_0 = 7.0, 9.0$ (dB), predicted by generating functional analysis (GFA) and density evolution (DE) [6], where E_b/N_0 (dB) denotes $10 \log_{10} E_b/N_0$ (see the appendix). The system load is $\beta = 0.5 < \beta_c$, where β_c is the critical system load defined as the minimum system load at which the dynamics fail to converge. Figure 3 shows the first few stages of the detection dynamics obtained from 100 experiments for the cases $E_b/N_0 = 5.5$, 7.5 (dB), predicted by GFA and DE with the system load $\beta = 0.7 > \beta_c$. Oscillation of the detection dynamics was observed, when $\beta > \beta_c$. In such a case, both GFA and DE predicted the failure of convergence of the dynamics. However, the DE results has residual deviations in figures 2 and 3 due to the lack of the Onsager reaction term and the assumption that the local field follows a Gaussian distribution. In particular, the deviation of the DE predictions from the simulation results becomes large when $\beta > \beta_c$. In contrast, GFA exhibits good consistency with the simulation results for any system load.

The difference between DE and GFA also appears in a signal term with respect to the information bit of the local field. The signal terms of DE and GFA at stage t represent B_t

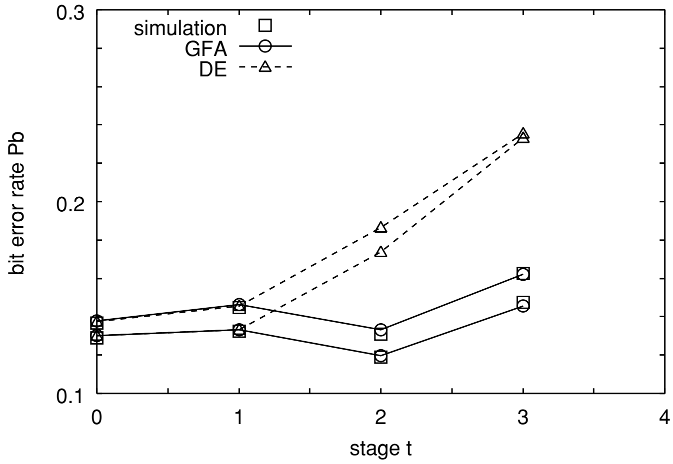

Figure 3. The first few stages of the detection dynamics predicted by generating functional analysis (solid line) and density evolution (dashed line). Computer simulations (square) are evaluated with $N = 8000$ from 100 experiments for the cases $E_b/N_0 = 5.5$ (dB) (upper), 7.5 (dB) (lower). The system load is $\beta = 0.7 > \beta_c$ for both cases.

and $\hat{k}(t)$, respectively (see the appendix). The signal term $\hat{k}(t)$ derived by GFA contains all response functions $G(s, s')$ with $s, s' \leqslant t$. On the other hand, the signal term B_t derived by DE contains only the response functions of adjacent stages. This difference appears from stage $t = 1$. The signal term $\hat{k}(1)$ of GFA is

$$\hat{k}(1) = \begin{vmatrix} 1 & \beta G(0,-1) & \beta G(1,-1) \\ 0 & 1 & \beta G(1,0) \\ 1 & 1 & 1 \end{vmatrix} = 1 - \beta G(1,0) + \beta^2 G(1,0)G(0,-1) - \beta G(1,-1), \tag{61}$$

while the signal term B_1 of DE is

$$B_1 = 1 - \beta U_1 + \beta^2 U_1 U_0. \tag{62}$$

As you can easily see, B_1 contains only U_1 and U_0, which correspond to $G(1, 0)$ and $G(0, -1)$ of GFA respectively, while the $\hat{k}(1)$ has the response function between stage 1 and stage -1 as $G(1, -1)$.

5. Conclusions

We presented the generating functional analysis to describe the detection dynamics of PIC for CDMA multiuser detection. The predictions of DE can qualitatively explain the detection dynamics only when the detection dynamics converge. Furthermore, the deviation of the predictions of DE from the results of numerical experiments becomes large when the detection dynamics fail to convergence. In contrast, the predictions of GFA are in good agreement with computer simulation result of PIC for any system load and channel noise level, even if the dynamics fail to converge.

Acknowledgments

This work was partially supported by a Grant-in-Aid for Scientific Research on Priority Areas no. 14084212 and for Scientific Research (C) no. 16500093 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Appendix. Density evolution of CDMA detection dynamics

Density evolution is a useful tool to analyse nonlinear dynamics [7, 8]. By means of density evolution, the bit error rate $P_b(t)$ of hard decisions $\hat{b}_k(t) = \mathrm{sgn}[u_k(t-1)]$ at the tth stage is given by

$$P_b(t) = \frac{1 - M_t}{2}, \tag{A.1}$$

where M_t are to be evaluated by the following recursive formulae for B_t, $C_{t,\tau}$, M_t, U_t and $q_{t,\tau}$:

$$B_t = 1 - \beta U_t B_{t-1}, \tag{A.2}$$

$$C_{t,\tau} = V_{t,\tau} + \beta^2 U_t U_\tau C_{t-1,\tau-1} + \sum_{\lambda=-1}^{t-1} V_{\lambda,\tau} \prod_{\kappa=\lambda+1}^{t} (-\beta U_\kappa) + \sum_{\lambda=-1}^{\tau-1} V_{\lambda,t} \prod_{\kappa=\lambda+1}^{\tau} (-\beta U_\kappa), \tag{A.3}$$

$$V_{t,\tau} = \sigma_0^2 + \beta(1 - M_t - M_\tau + q_{t,\tau}), \tag{A.4}$$

$$M_{t+1} = \int Dz\, \mathrm{sgn}(B_t + z\sqrt{C_{t,t}}), \tag{A.5}$$

$$U_{t+1} = \frac{1}{\sqrt{C_{t,t}}} \int Dz\, z\, \mathrm{sgn}(B_t + z\sqrt{C_{t,t}}), \tag{A.6}$$

$$\begin{aligned} q_{t+1,\tau+1} = \int\int\int Dz\, Du\, Dv\, &\mathrm{sgn}(B_t + z\sqrt{C_{t,\tau}} + u\sqrt{C_{t,t} - C_{t,\tau}}) \\ &\times \mathrm{sgn}(B_\tau + z\sqrt{C_{t,\tau}} + v\sqrt{C_{t,t} - C_{t,\tau}}), \end{aligned} \tag{A.7}$$

where $Dz \equiv (2\pi)^{-1/2}\, \mathrm{e}^{-z^2/2}\, \mathrm{d}z$. The initializations are $V_{-1,t} = V_{t,-1} = \sigma_0^2 + \beta(1 - M_t)$, $B_{-1} = 1$, $M_{-1} = 0$, $C_{-1,-1} = \sigma_0^2 + \beta$, $C_{-1,t} = C_{t,-1} = V_{-1,t} - \beta U_t V_{-1,t-1}$ and $q_{-1,t} = q_{t,-1} = 0$. The physical meaning of the parameters B_t, $C_{t,\tau}$, M_t, U_t and $q_{t,\tau}$ is

$$B_t = \mathrm{E}[u_k(t)], \tag{A.8}$$

$$C_{t,\tau} = \mathrm{Cov}[u_k(t), u_k(\tau)], \tag{A.9}$$

$$M_{t+1} = \frac{1}{K} \sum_{k=1}^{K} \mathrm{sgn}[u_k(t)], \tag{A.10}$$

$$U_{t+1} = \frac{1}{K} \sum_{k=1}^{K} \mathrm{sgn}'[u_k(t)], \tag{A.11}$$

$$q_{t+1,\tau+1} = \frac{1}{K} \sum_{k=1}^{K} \mathrm{sgn}[u_k(t)]\, \mathrm{sgn}\,[u_k(\tau)]. \tag{A.12}$$

The detailed derivation is available in the appendix of the paper [6]. In the derivation by means of density evolution, it is assumed that the local field $u_k(t)$ follows the Gaussian distribution with mean B_t and covariance $C_{t,\tau}$. Furthermore, the Onsager reaction term is ignored. The signal term B_t contains only the response functions of adjacent stages.

References

[1] Varanasi M K and Aazhang 1990 *IEEE Trans. Commun.* **38** 509
[2] Tanaka T 2001 *Europhys. Lett.* **54** 540
[3] Tanaka T 2002 *IEEE Trans. Inf. Theory* **48** 2888
[4] Nishimori H 2002 *Europhys. Lett.* **57** 302
[5] Kabashima Y 2003 *J. Phys. A: Math. Gen.* **36** 11111
[6] Tanaka T and Okada M 2005 *IEEE Trans. Inf. Theory* **51** 700
[7] Okada M 1995 *Neural Netw.* **8** 833
[8] Richardson T J and Urbanke R L 2001 *IEEE Trans. Inf. Theory* **47** 599
[9] Coolen A C C 2000 *Preprint* cont-mat/0006011
[10] Düring A, Coolen A C C and Sherrington D 1998 *J. Phys. A: Math. Gen.* **31** 8609
[11] Kawamura M and Okada M 2002 *J. Phys. A: Math. Gen.* **35** 253
[12] Mimura K, Kawamura M and Okada M 2004 *J. Phys. A: Math. Gen.* **37** 6437